L^∞-ERROR ESTIMATE FOR THE FINITE ELEMENT METHOD ON TWO DIMENSIONAL SURFACES

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Abstract. We approximate the solution of the equation

$$-\Delta_S u + u = f \tag{0.1}$$

on a two-dimensional, embedded, orientable, closed surface S where $-\Delta_S$ denotes the Laplace Beltrami operator on S by using continuous, piecewise linear finite elements on a triangulation of S with flat triangles. We show that the L^∞-error is of order $O(h^2|\log h|)$ as in the corresponding situation in an Euclidean setting.

Key words. linear elliptic equation; two-dimensional surface; finite elements

1. Introduction. During the last years several articles appeared which deal with the numerical solution of linear partial differential equations which are defined on a hypersurface in $\mathbb{R}^3$. Roughly spoken their common aim is to show that concepts and properties which are well-known in an Euclidean setting carry over to the surface case. Without claiming completeness we summarize some steps towards this goal.

In [6] the finite element approximation of the Laplace-Beltrami equation

$$-\Delta_S u = f \tag{1.1}$$

on a surface S with continuous, piecewise linear elements (on a polyhedral approximation of S) is presented and it is shown that the L^2- and H^1- error estimates known from the corresponding Euclidean setting carry over to this case. In [7] this idea is extended to a semi-discrete approximation of linear parabolic equations which are defined on a (moving) hypersurface for which the motion is a priori given by a smooth one parameter family of diffeomorphisms of a fixed initial surface. Here, one has to take care of the fact that the time derivative is defined suitably which is tackled by the concept of the material derivative and the spatial discretization uses a moving mesh (the method is called ESFEM). Furthermore, $L^\infty(L^2)$- and $L^2(H^1)$-error estimates are shown and in [8] the $L^\infty(L^2)$-estimate is improved to the optimal order of $O(h^2)$. We refer to [9] for an survey of finite element methods for surface PDEs.

We mention further contributions to this topic in the literature. In [11] Runge-Kutta methods known from the semi-linear Euclidean setting, cf. [15, 16, 17], are adapted to ESFEM to obtain a fully discrete approximation of the linear parabolic equation in combination with the moving surface. See also [14] for a backwards difference time discretization of this problem. In [12] an additional tangential motion of the grid for ESFEM is introduced to improve the mesh quality, or more precisely, to compensate a motion related possibly deterioration of the mesh. In [22] finite element spaces that are induced by triangulations of an 'outer' domain are used to discretize partial differential equations on a surface, see also [20]. In [23, 21] an Eulerian finite element method for solving linear parabolic partial differential equations is presented and a stabilized finite element method for linear parabolic equations on surfaces is studied. In [10] a h-narrow band finite element method for linear elliptic equations on implicit surfaces is studied. See also [2] for variational problems and partial differential

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equations on implicit surfaces. In [4] an analysis of the discontinuous Galerkin method for linear elliptic problems on surfaces is carried out.

In our paper we show that the well-known L^∞-estimate for the finite element approximation of linear elliptic equations in a two dimensional Euclidean setting, cf. [24] and compare also [18, 19, 13], carries over to the case of a linear elliptic equation on a surface in $\mathbb{R}^3$ which seems to be omitted in the literature until now according to the author's knowledge.

We refer to [1] where an icosahedral discretization of the two-sphere is used to solve the Laplace-Beltrami equation on the two-sphere. There it is claimed (without detailed justification) that the quadratic order of the L^∞-interpolation error immediately carries over to an L^∞-estimate of quadratic order for the discretization error, see the passage following Table 2 on page 1114 in [1], which is wrong, of course.

Our paper is organized as follows. In the remaining part of the introduction we present the general setting and formulate our partial differential equation. In Section 2 we introduce our notation, state some basic facts which will be used in the sequel several times and present the discretization of the equation. In Section 3 we present for completeness in the surface case the known H^1- and L^2-error estimates, cf. [6]. In Section 4 we state in Theorem 4.1 our main result about the L^∞-error estimate in the surface case and present a proof.

We sketch the idea of the proof. We prove the estimate pointwise by using an approximative Green's function $\tilde{v}$ on the surface. The latter function is obtained by lifting a cutted-off Euclidean Green's function from the tangent plane to the surface at which the appearing discrepancy to an exact Green's function on the surface is – in case of relevance – supressed by the L^2-error estimates from Section 3. We define a finite element approximation of $\tilde{v}$ for which we prove an error estimate in the $W^{1,1}$-norm which has the same order as in the corresponding Euclidean case. In doing so we adapt the argumentation from [24].

Let S be a smooth two-dimensional, embedded, orientable, closed hypersurface in $\mathbb{R}^3$. We triangulate the surface by a family T_h of flat triangles with corners (i.e. nodes) lying on S. We denote the surface of class $C^{0,1}$ given by the union of the triangles $\tau \in T_h$ by S_h; the union of the corresponding nodes is denoted by N_h. Here, $h > 0$ denotes a discretization parameter which is related to the triangulation in the following way. For $\tau \in T$ we define the diameter $\rho(\tau)$ of the smallest disc containing τ, the diameter $\sigma(\tau)$ of the largest disc contained in τ and

$$h = \max_{\tau \in T_h} \rho(\tau), \quad \gamma_h = \min_{\tau \in T_h} \frac{\sigma(\tau)}{h}. \tag{1.2}$$

We assume that the family $(T_h)_{h>0}$ is quasi-uniform, i.e. $\gamma_h \geq \gamma_0 > 0$. We let

$$V_h = \{v \in C^0(S_h) : v_{|\tau} \text{ linear for all } \tau \in T_h\} \tag{1.3}$$

be the space of continuous piecewise linear finite elements.

We assume $f \in L^2(S)$ and our goal is to prove error estimates for a finite element approximation of the unique solution $u \in H^2(S)$ of the PDE

$$-\Delta_S u + u = f \tag{1.4}$$

where Δ_S is the Laplace-Beltrami operator on S. In Section 4 we will assume that f is in addition so that $u \in W^{2,\infty}(S)$.

REMARK 1.1. *After submitting a first version of the present article to arXiv the author became aware of the fact that Theorem 4.1 has been proved in [5].*

2. Notations, elementary observations and discrete formulation. Let N be a tubular neighborhood of S in which the Euclidean metric of N can be written in the coordinates $(x^0, x) = (x^0, x^i)$ of the tubular neighborhood as

$$\bar{g}_{\alpha\beta} = (dx^0)^2 + \sigma_{ij}(x) dx^i dx^j. \tag{2.1}$$

Here, x^0 denotes the globally (in N) defined signed distance to S and $x = (x^i)_{i=1,2}$ local coordinates for S.

For small h we can write S_h as graph (with respect to the coordinates of the tubular neighborhood) over S, i.e.

$$S_h = \operatorname{graph} \psi = \{(x^0, x) : x^0 = \psi(x), x \in S\} \tag{2.2}$$

where $\psi = \psi_h \in C^{0,1}(S)$ suitable. Note, that

$$|D\psi|_\sigma \le ch, \quad |\psi| \le ch^2. \tag{2.3}$$

The induced metric of S_h is given by

$$g_{ij}(\psi(x), x) = \frac{\partial \psi}{\partial x^i}(x) \frac{\partial \psi}{\partial x^j}(x) + \sigma_{ij}(x). \tag{2.4}$$

Hence we have for the metrics, their inverses and their determinants

$$g_{ij} = \sigma_{ij} + O(h^2), \quad g^{ij} = \sigma^{ij} + O(h^2) \quad \text{and} \quad g = \sigma + O(h^2)|\sigma_{ij}\sigma^{ij}|^{\frac{1}{2}} \tag{2.5}$$

where we use summation convention.

Let $f \in W^{1,p}(S)$, $g \in W^{1,p^*}(S)$, $1 \le p \le \infty$ and p^* Hölder conjugate of p. We define the so-called lift $\tilde{f}$ of f to S_h by $f(x) = \tilde{f}(\psi(x), x)$, $x \in S$, and correspondingly for g (more generally, we can do this procedure whenever we have two graphs in the same coordinate system and denote it by the terminus lift, furthermore, this terminus can be obviously extended to subsets). In local coordinates $x = (x^i)$ of S hold

$$\int_S \langle Df, Dg \rangle = \int_S \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \sigma^{ij}(x) \sqrt{\sigma(x)} dx^i dx^j, \tag{2.6}$$

$$\int_{S_h} \left\langle D\tilde{f}, D\tilde{g} \right\rangle = \int_S \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} g^{ij}(\psi(x), x) \sqrt{g(\psi(x), x)} dx^i dx^j, \tag{2.7}$$

$$\int_S \langle Df, Dg \rangle = \int_{S_h} \left\langle D\tilde{f}, D\tilde{g} \right\rangle + O(h^2) \|f\|_{W^{1,p}(S)} \|g\|_{W^{1,p^*}(S)}, \tag{2.8}$$

and similarly,

$$\int_S f = \int_{S_h} \tilde{f} + O(h^2) \|f\|_{L^1(S)} \tag{2.9}$$

where now $f \in L^1(S)$ is sufficient.

The bracket $\langle u, v \rangle$ denotes here the scalar product of two tangent vectors u, v (or their covariant counterparts), and later also the application of a distribution u to a test function v; which meaning is on hand will be clear from context. $\|\cdot\|_{W^{k,p}}$ denotes the usual Sobolev norm, $|\cdot|_{W^{k,p}} = \sum_{|\alpha|=k} \|D^\alpha \cdot\|_{L^p}$ and $H^k = W^{k,2}$.

Remark 2.1. *Let $z_0 \in S$ and $T_{z_0}S$ be the tangent plane of S in z_0. A ball $B_{2r_1}(z_0) \subset S$, $r_1 > 0$ suitable, and a corresponding portion of S_h can be written*

as a graph over a corresponding subset U of $T_{z_0}S$ in a (perpendicular) Euclidean coordinate system (x^0, x^1, x^2). Here, x^1, x^2 denote Euclidean coordinates in $T_{z_0}S$ with center in z_0 and x^0 denotes the coordinate axis perpendicular to $T_{z_0}S$ so that $T_{z_0}S = \{x^0 = 0\}$. One can consider lifts of functions between these three (pieces of) surfaces with respect to this Euclidean representation as graph. Analogous estimates to (2.9), (2.8) hold except for lifts from or to U, for these $O(h^2)$ has to be replaced by $O(\max(\operatorname{diam}(\operatorname{supp} f), \operatorname{diam}(\operatorname{supp} g))^2)$. Since S is compact all constants in the estimates and r_1 can be chosen independently from z_0.

Since the properties and aspects needed to prove a priori error estimates for finite element approximations are formulated in terms of integrals these observations concerning the transformation behavior of integrals essentially imply that the known error estimates from the Euclidean setting carry over to the surface case as far as convergence of at most quadratic order is concerned.

We define

$$a : W^{1,p}(S) \times W^{1,p^*}(S) \to \mathbb{R}, \quad a(u, v) = \int_S \langle Du, Dv\rangle + uv \tag{2.10}$$

and

$$a_h : W^{1,p}(S_h) \times W^{1,p^*}(S_h) \to \mathbb{R}, \quad a(u_h, v_h) = \int_{S_h} \langle Du_h, Dv_h\rangle + u_h v_h. \tag{2.11}$$

The finite element approximation of u in (1.4) is defined as the unique $u_h \in V_h$ with

$$a_h(u_h, \varphi_h) = \int_{S_h} f_h \varphi_h \quad \forall \varphi_h \in V_h \tag{2.12}$$

where f_h is the lift of f to S_h. Note, that existence follows from uniqueness.

Constants that do not depend on h or z_0 are denoted by c or c_0, c_1 if several appear and should be specifiable.

Functions labelled by capital letters denote (without explicit declaration) the lift via the representation as graph with respect to the tubular neighborhood N of S of the function denoted with the corresponding small letter to the other surface, e.g. assume w is defined on S then W denotes the lift of w to S_h and vice versa, i.e. if w is defined on S_h then W denotes the lift of w to S.

3. The H^1-estimate and L^2-estimate. For completeness we give in this Section a proof of the well-known estimates stated in Lemma 3.1 and Lemma 3.2, cf. [6].

LEMMA 3.1. *We have*

$$\|u - U_h\|_{H^1(S)} \le O(h)\|f\|_{L^2(S)}. \tag{3.1}$$

Proof. Let $\varphi_h \in V_h$ arbitrary then

$$\begin{aligned} \|u - U_h\|^2_{H^1(S)} =& a(u - U_h, u - U_h) \\ =& a(u - U_h, u - \Phi_h) + a(u - U_h, \Phi_h - U_h) \\ \le& \|u - U_h\|_{H^1(S)}\|u - \Phi_h\|_{H^1(S)} + a(u - U_h, \Phi_h - U_h). \end{aligned} \tag{3.2}$$

We rewrite

$$a(u-U_h,\Phi_h-U_h)=\int_S f(\Phi_h-U_h)-\int_{S_h} f_h(\varphi_h-u_h) \tag{3.3}$$
$$+O(h^2)\|U_h\|_{H^1(S)}\|\Phi_h-U_h\|_{H^1(S)}$$
$$=O(h^2)(\|U_h\|_{H^1(S)}+\|f\|_{L^2})\|\Phi_h-U_h\|_{H^1(S)}$$

so that we obtain from (3.2)

$$\|u-U_h\|_{H^1(S)}\le 2\max(m_1,m_2) \tag{3.4}$$

where

$$m_1=\inf_{\varphi_h\in V_h}\|u-\Phi_h\|_{H^1(S)} \tag{3.5}$$

and

$$m_2=\left\{O(h^2)\|f\|_{L^2(S)}(\inf_{\varphi_h\in V_h}\|\Phi_h-u\|_{H^1(S)}+\|u-U_h\|_{H^1(S)})\right\}^{\frac{1}{2}}. \tag{3.6}$$

Let $\tilde u$ be the lift of u to S_h then

$$\inf_{\varphi_h\in V_h}\|u-\Phi_h\|_{H^1(S)}\le\|u-\tilde u\|_{H^1(S)}+\inf_{\varphi_h\in V_h}\|\tilde u-\varphi_h\|_{H^1(S_h)}+O(h^2)\|f\|_{L^2(S)} \tag{3.7}$$
$$\le O(h)\|f\|_{L^2(S)}.$$

Putting these estimates together yields the claim. □

The estimate in the L^2-norm can be improved.

Lemma 3.2. *We have*

$$\|u-U_h\|_{L^2(S)}\le O(h^2)\|f\|_{L^2(S)}. \tag{3.8}$$

Proof. Let $w\in H^2(S)$ be the unique solution of $-\Delta_S w+w=u-U_h$ and $w_h\in V_h$ the corresponding unique finite element solution to the right-hand side $\tilde u-u_h$. Then we have

$$\int_S(u-U_h)^2=a(u-U_h,w) \tag{3.9}$$
$$=a(u-U_h,w-W_h)+a(u-U_h,W_h)$$
$$\le\|u-U_h\|_{H^1(S)}\|w-W_h\|_{H^1(S)}$$
$$+O(h^2)(\|U_h\|_{H^1(S)}+\|f\|_{L^2(S)})\|W_h\|_{H^1(S)}$$
$$\le ch\|u-U_h\|_{H^1(S)}\|u-U_h\|_{L^2(S)}$$
$$+O(h^2)(\|U_h\|_{H^1(S)}+\|f\|_{L^2(S)})\|u-U_h\|_{L^2(S)}.$$

□

4. The L^∞-estimate. We assume that $f\in L^2(S)$ is in addition so that $u\in W^{2,\infty}(S)$. The following theorem states our main result.

Theorem 4.1. *There holds*

$$\|u-U_h\|_{L^\infty(S)}\le ch^2|\log h|\|u\|_{W^{2,\infty}(S)}. \tag{4.1}$$

The proof of the corresponding Euclidean statement is well-known, cf. [24].

The purpose of the remaining part of this section is to prove Theorem 4.1.

Let $z_0 \in S$ and

$$\varphi_S : U \to B_{2r_1}(z_0), \quad (0,x) \mapsto (\psi_S(x), x), \quad x = (x^1, x^2) \tag{4.2}$$

be the representation as graph of $B_{2r_1}(z_0) \subset S$ over $U \subset T_{z_0}S$ according to Remark 2.1.

DEFINITION 4.2. *We set $B_j = B_{jr_1}(z_0)$, $j = 1, 2$, $\varphi = \varphi_S$ and let v be the Euclidean Green's function with respect to $-\Delta + I$ in $T_{z_0}S \equiv \mathbb{R}^2$ with singularity in z_0, i.e., more precisely,*

$$\begin{aligned} -\Delta v + v =&\delta_{z_0} \quad \text{in } B_{100}(z_0) \subset \mathbb{R}^2 \\ \partial_n v =&0 \quad \text{on } \partial B_{100}(z_0). \end{aligned} \tag{4.3}$$

Let $\zeta \in C_0^\infty(B_{\frac{3}{2}r_1}(z_0))$, $\zeta \equiv 1$ in B_1, be a cut-off function and set

$$\tilde{v}(x) = v(\varphi^{-1}(x))\zeta(x), \quad \tilde{l}(x) = l(\varphi^{-1}(x))\zeta(x), \quad x \in B_2, \tag{4.4}$$

where $l(z) = \log|z - z_0|$.

There holds

$$v - \frac{1}{2\pi} l \in H^2(B_{100}(z_0)), \tag{4.5}$$

cf. [24, Lemma 1, page 687]. W.l.o.g. we may assume $\|\tilde{v}\|_{C^2(B_2 \setminus B_1)} \le c_0$ and

$$\|\tilde{v} - \frac{1}{2\pi}\tilde{l}\|_{W^{2,2}(S)} \le c \tag{4.6}$$

where c_0, c are independent from z_0.

The next Lemma shows in which sense $\tilde{v}$ is an approximative Green's function.

LEMMA 4.3. *Let $\tilde{v}$ be as in Definition 4.2. Let $w \in H^1(S) \cap C^0(S)$ then*

$$|\langle -\Delta_S \tilde{v} + \tilde{v}, w\rangle - w(z_0)| \le c\|w\|_{L^2(S)}. \tag{4.7}$$

Proof. Let $\eta \in C_0^\infty(B_{r_1}(z_0))$, $\eta_{|B_{r_1/2}(z_0)} \equiv 1$, $|\eta| \le c$ and $|D\eta| \le c$. We write

$$w = \eta w + (1 - \eta)w = w^1 + w^2 \tag{4.8}$$

and have (in local coordinates induced from φ)

$$\begin{aligned} \langle -\Delta_S \tilde{v} + \tilde{v}, w\rangle =& \left\langle -\Delta_S \tilde{v} + \tilde{v}, w^1\right\rangle + \left\langle -\Delta_S \tilde{v} + \tilde{v}, w^2\right\rangle \\ =& \int_U \left(g^{ij} \frac{\partial \tilde{v}}{\partial x^i} \frac{\partial w^1}{\partial x^j} + \tilde{v} w^1 \right) \sqrt{g} + \int_U \left(g^{ij} \frac{\partial \tilde{v}}{\partial x^i} \frac{\partial w^2}{\partial x^j} + \tilde{v} w^2 \right) \sqrt{g}. \end{aligned} \tag{4.9}$$

We rewrite the second integral on the right-hand side of (4.9) as

$$\begin{aligned} &\int (\tilde{v} - \delta^{ij} \frac{\partial^2 \tilde{v}}{\partial x^i \partial x^j}) \sqrt{g} w^2 - \int \frac{\partial}{\partial x^j} g^{ij} \frac{\partial \tilde{v}}{\partial x^i} \sqrt{g} w^2 \\ &\quad - \int g^{ij} \frac{\partial \tilde{v}}{\partial x^i} \frac{1}{2} \sqrt{g} g^{kl} \frac{\partial}{\partial x^j} g_{kl} w^2 + \int (\delta^{ij} - g^{ij}) \frac{\partial^2 \tilde{v}}{\partial x^i \partial x^j} \sqrt{g} w^2 \\ &= O(1)\|w\|_{L^2(S)} \end{aligned} \tag{4.10}$$

where we used integration by parts and all integrals are over $U \setminus \varphi^{-1}(B_{\frac{r_1}{2}}(z_0))$.

Let r denote the distance to z_0 in $T_{z_0}S$ then we obtain

$$
\begin{aligned}
|\delta^{ij} - g^{ij}| \le & cr^2, \quad |g^{ij}| \le c, \quad \left|\frac{\partial}{\partial x^k} g_{ij}\right| \le cr, \\
& \left|\frac{\partial \tilde{l}}{\partial x^i}\right| \le \frac{c}{r}, \quad \left|\frac{\partial^2 \tilde{l}}{\partial x^i \partial x^j}\right| \le \frac{c}{r^2}.
\end{aligned}
\tag{4.11}
$$

We rewrite the first integral on the right-hand side of (4.9) as

$$
\begin{aligned}
&\int_U \left(g^{ij}\frac{\partial \tilde{v}}{\partial x^i}\frac{\partial w^1}{\partial x^j} + \tilde{v}w^1\right)\sqrt{g} \\
=&\int_U \left(g^{ij}\frac{\partial \tilde{v}}{\partial x^i}\frac{\partial w^1}{\partial x^j} + \tilde{v}w^1\right)(\sqrt{g}-1) + \int_U (g^{ij}-\delta^{ij})\frac{\partial \tilde{v}}{\partial x^i}\frac{\partial w^1}{\partial x^j} \\
&+ \int_U \left(\delta^{ij}\frac{\partial \tilde{v}}{\partial x^i}\frac{\partial w^1}{\partial x^j} + \tilde{v}w^1\right) \\
=&w(z_0) + O(1)\|w\|_{L^2(S)}
\end{aligned}
\tag{4.12}
$$

where we used (4.11), Hölder's inequality, (4.3) and that we are allowed to perform integration by parts in the integrals with the factors $(\sqrt{g}-1)$ and $g^{ij}-\delta^{ij}$, note, that

$$
\begin{aligned}
&\int_U \left|(\delta^{ij}-g^{ij})\frac{\partial^2 \tilde{v}}{\partial x^i \partial x^j} w^1\right| \\
\le &\int_U \left|(\delta^{ij}-g^{ij})\left(|\frac{\partial^2(\tilde{v}-\tilde{l})}{\partial x^i \partial x^j}| + |\frac{\partial^2 \tilde{l}}{\partial x^i \partial x^j}|\right) w^1\right| \\
\le & c\|w\|_{L^2(S)}.
\end{aligned}
\tag{4.13}
$$

□

Remark 4.4. *From now, we denote the approximative Green's function $\tilde{v}$ by g (there will be no ambiguity with the symbol for the determinant of the metric).*

We define an approximation $g_h \in V_h$ of g by

$$
a_h(g_h, v_h) = a(g, V_h)
\tag{4.14}
$$

for all $v_h \in V_h$.

Lemma 4.5. *Assume*

$$
\|g - G_h\|_{W^{1,1}(S)} \le ch|\log h|
\tag{4.15}
$$

then Theorem 4.1 follows.

Proof. From Lemma 4.3 we conclude

$$
\begin{aligned}
(u-U_h)(z_0) =&a(g, u-U_h) + O(h^2)\|f\|_{L^2(S)} \\
=&a_h(G-g_h, U-u_h) + O(h^2)\|g\|_{W^{1,1}(S)}\|u-U_h\|_{W^{1,\infty}(S)} \\
&+ a_h(g_h, U-u_h).
\end{aligned}
\tag{4.16}
$$

We estimate

$$
\begin{aligned}
a_h(g_h, U-u_h) =&a(G_h, u) - a_h(g_h, u_h) + O(h^2)\|G_h\|_{W^{1,1}(S)}\|u\|_{W^{1,\infty}(S)} \\
=&O(h^2)\left(\|G_h\|_{W^{1,1}(S)}\|u\|_{W^{1,\infty}(S)} + \|G_h\|_{L^1(S)}\|u\|_{W^{2,\infty}(S)}\right).
\end{aligned}
\tag{4.17}
$$

Furthermore, we have

$$a_h(G-g_h,U-u_h)=a_h(G-g_h,U-v_h)+a_h(G-g_h,v_h-u_h), \tag{4.18}$$

rewrite the second summand by using Lemma 4.3 as

$$\begin{aligned} &a_h(G-g_h,v_h-u_h)\\ &\quad=a(g,V_h-U_h)+O(h^2)\|g\|_{W^{1,1}(S)}\|V_h-U_h\|_{W^{1,\infty}(S)}\\ &\qquad-a_h(g_h,v_h-u_h)\\ &\quad\le O(h^2)\|g\|_{W^{1,1}(S)}\|V_h-U_h\|_{W^{1,\infty}(S)} \end{aligned} \tag{4.19}$$

and estimate the first summand as follows

$$|a_h(G-g_h,U-v_h)|\le\|g-G_h\|_{W^{1,1}(S)}\|u-V_h\|_{W^{1,\infty}(S)} \tag{4.20}$$

We let $v_h\in V_h$ be the interpolation of u and obtain the claim from

$$\begin{aligned} \|u-U_h\|_{W^{1,\infty}(S)}\le&\|u-V_h\|_{W^{1,\infty}(S)}+\|V_h-U_h\|_{W^{1,\infty}(S)}\\ \le&\|u-v_h\|_{W^{1,\infty}(S)}\\ &+ch^{-1}\left(\|V_h-u\|_{W^{1,2}(S)}+\|u-U_h\|_{W^{1,2}(S)}\right) \end{aligned} \tag{4.21}$$

which holds in view of estimate (4.37) . □

In order to show (4.15) we prove as first step the following Lemma.

Lemma 4.6. *We have*

$$\|g-G_h\|_{L^2(S)}\le ch \tag{4.22}$$

where c is independent of z_0.

Proof. Let τ be a triangle in T_h containing the lift $\tilde z_0$ of z_0, and let q be the linear function with

$$\int_\tau qp=p(z_0) \tag{4.23}$$

for all linear functions p. Because τ contains a disk of radius $\gamma_0 h$ we see that

$$\sup_\tau|q|\le ch^{-2}. \tag{4.24}$$

We extend the domain of definition of q to S_h by zero and set $\tilde\delta=Q$.

We define

$$\psi(v)=a(g,v)-\langle\delta_{z_0},v\rangle,\quad v\in W^{1,\infty}(S)\cap C^0(S), \tag{4.25}$$

From Lemma 4.3 we deduce that

$$|\psi(v)|\le c_0\|v\|_{L^2(S)} \tag{4.26}$$

so that by Hahn-Banach Theorem ψ can be extended to a linear functional on $L^2(S)$– denoted by ψ as well – with norm $\le c_0$. W.l.o.g we may assume that $\psi\in L^2(S)$.

Let $w\in H^1(S)$ and $w_h\in V_h$ be the solutions of

$$\begin{aligned} a(w,v)&=\int_S\psi v\quad\forall v\in H^1(S)\\ a_h(w_h,v_h)&=\int_{S_h}\Psi v_h\quad\forall v_h\in V_h. \end{aligned} \tag{4.27}$$

Lemma 3.1 leads to

$$\|w - W_h\|_{H^1(S)} \le ch \tag{4.28}$$

where c independent from z_0.

Let $z_h \in V_h$ with

$$a_h(z_h, v_h) = \langle \delta_{\tilde{z}_0}, v_h \rangle = v_h(\tilde{z}_0). \tag{4.29}$$

Let $\tilde{g}$ solve

$$-\Delta_S \tilde{g} + \tilde{g} = \tilde{\delta}. \tag{4.30}$$

Since z_h can be seen as finite element approximation of $\tilde{g}$ we have in view of Lemma 3.1

$$\|\tilde{g} - Z_h\|_{H^s(S)} \le ch^{2-s}\|\tilde{\delta}\|_{L^2(S)} \le ch^{1-s}, \quad s = 0, 1. \tag{4.31}$$

In view of

$$\begin{aligned} a_h(g_h - w_h, v_h) =& a(g, V_h) - \int_{S_h} \Psi v_h \\ =& a(g, V_h) - \int_S \psi V_h + O(h^2)\|\psi\|_{L^2(S)}\|V_h\|_{L^2(S)} \\ =& \langle \delta_{z_0}, V_h \rangle + O(h^2)\|\psi\|_{L^2(S)}\|V_h\|_{L^2(S)}, \quad \forall v_h \in V_h \end{aligned} \tag{4.32}$$

we deduce

$$\|z_h - (g_h - w_h)\|_{L^2(S)} \le O(h^2). \tag{4.33}$$

To estimate

$$\|g - G_h\|_{L^2(S)} = \|(g - w - \tilde{g}) + (\tilde{g} - Z_h) + (w - W_h)\|_{L^2(S)} + O(h^2) \tag{4.34}$$

we need to estimate $\|g - w - \tilde{g}\|_{L^2(S)}$. Let $\varphi \in C^\infty(S)$ and $\tilde{w}$ a solution of

$$-\Delta_S \tilde{w} + \tilde{w} = \varphi \tag{4.35}$$

then

$$\begin{aligned} \int_S (g - w - \tilde{g})\varphi =& a(g - w - \tilde{g}, \tilde{w}) \\ =& \left\langle \delta_{z_0} - \tilde{\delta}, \tilde{w} \right\rangle \\ =& \left\langle \delta_{z_0} - \tilde{\delta}, \tilde{w} - \tilde{w}_I \right\rangle \\ \le& \|\tilde{w} - \tilde{w}_I\|_{L^\infty(S)} + \|\tilde{\delta}\|_{L^2(S)}\|\tilde{w} - \tilde{w}_I\|_{L^2(S)} \\ \le& O(h)\|\varphi\|_{L^2(S)}. \end{aligned} \tag{4.36}$$

Here, $\tilde{W}_I$ denotes the linear interpolation of $\tilde{w}$, $\tilde{w}_I$ its lift to S and we used, cf. [3, Theorem 4.4.20],

$$h^{\frac{2}{p}}\|\chi - \chi_I\|_{L^\infty(S)} + \sum_{j=0}^{1} h^j\|\chi - \chi_I\|_{W^{j,p}(S)} \le ch^2\|\chi\|_{W^{2,p}(S)}, \quad 1 \le p \le \infty, \tag{4.37}$$

for $\chi \in H^2(S)$ and χ_I the linear interpolation of χ (, and the right-hand side possibly unbounded). ☐

Remark 4.7. *(i) Estimate (4.15) follows immediately if we show*

$$\|\tilde{l} - \tilde{L}_h\|_{W^{1,1}(S)} \le ch|\log h| \tag{4.38}$$

where $\tilde{l}_h \in V_h$ is defined by

$$a_h(\tilde{l}_h, v_h) = a(\tilde{l}, V_h) \quad \forall v_h \in V_h. \tag{4.39}$$

(ii) There holds

$$\|\tilde{l} - \tilde{L}_h\|_{L^2(S)} \le ch. \tag{4.40}$$

Proof. (i) We have

$$a_h(g_h - \frac{1}{2\pi}\tilde{l}_h, v_h) = a(g - \frac{1}{2\pi}\tilde{l}, V_h) \tag{4.41}$$

so that in view of (4.5) we conclude from Lemma 3.1 that

$$\|G_h - \frac{1}{2\pi}\tilde{L}_h - (g - \frac{1}{2\pi}\tilde{l})\|_{H^1(S)} \le ch \tag{4.42}$$

and the triangle inequality implies (4.15).

(ii) Use (4.42), the triangle inequality and Lemma 4.6. ☐

In the remaining part of this section we prove (4.38). We recall that $l(z) = \log|z - z_0|$ is defined in $T_{z_0}S \equiv \mathbb{R}^2$, that r denotes the distance to z_0 in $T_{z_0}S$ and state that l has bounded mean oscillation in the following sense.

Lemma 4.8. *Let $z_1 \in \mathbb{R}^2$ and $0 < \rho < \infty$. Then there is a constant $l_0 \in \mathbb{R}$ depending on z_1 and ρ such that*

$$\int_{\{|z-z_1|\le\rho\}} (l - l_0)^2 \le 9\pi\rho^2. \tag{4.43}$$

Proof. This is the assertion of [24, Lemma 2 on page 688]. ☐

Remark 4.9. *In the following we will consider lifts of objects defined on $B_{2r_1}(z_0) \subset S$, $U \subset T_{z_0}S$ or a suitable portion of S_h to another of these three surfaces with respect to the representation as graph over U in (perpendicular) Euclidean coordinates as described in Remark 2.1. By adding the superscripts S, T or h we indicate to which surface the object is lifted, e.g. let $M \subset B_{2r_1}(z_0) \subset S$ then M^T denotes its lift to $T_{z_0}S$. Similar correction terms as in (2.8) and (2.9) appear when we lift integrands of (with a power of r) weighted $W^{1,p}$-norms, i.e. if we estimate such a norm then the lift produces (at most) a constant as factor on the right-hand side of the estimates.*

We estimate the error $E = \tilde{l} - \tilde{L}_h$ near z_0.

Lemma 4.10. *Let $0 < \rho < c_1 h$ be given and $B = \{|z - z_0| \le \rho\}$ a ball in $T_{z_0}S$. Then*

$$\int_{S\cap B^S} |\varphi_{z_0}^{-1}(z) - z_0|^\beta |DE|^p \le c\rho^\beta h^{2-p} \tag{4.44}$$

for $1 \le p < \beta + 2$.

Proof. We have $|D\tilde{l}| \leq \frac{c}{r}$ where $r = |\varphi_{z_0}^{-1}(z) - z_0|$. In view of Remark 4.9 we may w.l.o.g. consider r as a function as well on B^S and $B = B^T$ and get

$$\begin{aligned}\int_{S\cap B^S} |\varphi_{z_0}(z) - z_0|^\beta |DE|^p \leq & c\int_{S\cap B^S} r^\beta(|D\tilde{l}|^p + |D\tilde{L}_h|^p) \\ \leq & c\rho^{\beta+2-p} + \int_{B^h} r^\beta |D\tilde{l}_h|^p.\end{aligned} \tag{4.45}$$

By Lemma 4.8 there is $l_0 \in \mathbb{R}$ so that

$$\|l - l_0\|_{L^2(B^T)} \leq ch. \tag{4.46}$$

We get (using an inverse estimate to bound a $W^{1,\infty}$- by a L^2-norm)

$$\begin{aligned}\int_{B^h} r^\beta |D\tilde{l}_h|^p = & \int_{B^h} r^\beta |D(\tilde{l}_h - l_0)|^p \\ \leq & c\rho^\beta h^2 \sup_{B^h} |D(\tilde{l}_h - l_0)|^p \\ \leq & c\rho^\beta h^2 h^{-2p} \|\tilde{l}_h - l_0\|^p_{L^2(B^h)} \\ \leq & c\rho^\beta h^2 h^{-2p} (\|\tilde{L}_h - \tilde{l}\|_{L^2(B^S)} + \|l - l_0\|_{L^2(B^T)})^p \\ \leq & c\rho^\beta h^{2-p}\end{aligned} \tag{4.47}$$

in view of (4.40) and (4.46). ☐

Remark 4.11. *If we choose $\beta = 0$ in Lemma 4.10 we obtain that $\|DE\|_{L^1(S\cap B^S)} = O(h)$.*

We estimate the error E outside B^S, B as in Lemma 4.10, which means de facto in $B_2 \setminus B^S$ since $\operatorname{supp} \tilde{l} \subset B_{\frac{3}{2}r_0}(z_0)$ and get

$$\begin{aligned}\int_{B_2\setminus B^S} |DE| \leq & \left(\int_{B_2\setminus B^S} r^{-2}\right)^{\frac{1}{2}} \left(\int_S r^2|DE|^2\right)^{\frac{1}{2}} \\ \leq & c|\log h|^{\frac{1}{2}} \left(\int_S r^2|DE|^2\right)^{\frac{1}{2}}\end{aligned} \tag{4.48}$$

and

$$\begin{aligned}\int_S r^2|DE|^2 = & \int_S \langle DE, D(r^2E)\rangle - 2Er\langle DE, Dr\rangle \\ \leq & \int_S \langle DE, D(r^2E)\rangle + 2\left(\int_S E^2\right)^{\frac{1}{2}} \left(\int_S r^2|DE|^2\right)^{\frac{1}{2}}\end{aligned} \tag{4.49}$$

which leads by Peter-Paul inequality to

$$\int_S r^2|DE|^2 \leq 2\int_S \langle DE, D(r^2E)\rangle + 4\int_S E^2 \leq 2\int_S \langle DE, D(r^2E)\rangle + ch^2 \tag{4.50}$$

in view of Remark 4.7 (ii). The next goal is to show

$$\int_S \langle DE, D(r^2E)\rangle \leq \frac{1}{4}\int_S r^2|DE|^2 + ch^2|\log h| \tag{4.51}$$

which implies (4.38). Define

$$T^1 = \{\tau \in T_h : \text{dist } (\tilde{z}_0, \tau) \geq h\}, \quad \Omega_1 = \bigcup_{\tau \in T^1} \tau \subset S_h. \tag{4.52}$$

and note, that for small h

$$\{z \in S : \text{dist}_S(z, z_0) \geq 3h\} \subset \Omega_1^S \tag{4.53}$$

and $\tilde{l} \in C^\infty(\Omega_1^S)$. Let $\tilde{l}_I$ be a function in V_h which equals $\tilde{l}$ at all nodes in Ω_1. Let $\bar{l}_I$ denote the lift of $\tilde{l}_I$ to S.

Lemma 4.12. *There hold*

$$\int_{\Omega_1^S} (\bar{l}_I - \tilde{l})^2 \leq ch^2, \quad \int_{\Omega_1^S} r^{-2}|D(r^2(\bar{l}_I - \tilde{l}))|^2 \leq ch^2|\log h|. \tag{4.54}$$

Proof. Let $\tau \in T^1$ then

$$\begin{aligned} \|\tilde{l} - \bar{l}_I\|_{W^{s,\infty}(\tau^S)} \leq & ch^{2-s}\|\tilde{l}\|_{W^{2,\infty}(\tau^S)} \\ \leq & ch^{2-s}(\min_\tau r)^{-2} \quad s = 0, 1. \end{aligned} \tag{4.55}$$

Since $\min_\tau r \geq h$, $\max_\tau r - \min_\tau r \leq h$ we have

$$\frac{\max_\tau r}{\min_\tau r} \leq 2 \tag{4.56}$$

and hence for $\beta \geq 0$

$$\int_{\tau^S} r^\beta(\tilde{l} - \bar{l}_I)^2 + h^2 \int_{\tau^S} r^\beta |D(\tilde{l} - \bar{l}_h)|^2 \leq c \int_{\tau^S} r^\beta h^4 (\min_\tau r)^{-4} \leq c \int_{\tau^S} r^{\beta - 4} h^4. \tag{4.57}$$

Summing over all $\tau \in T^1$ implies the Lemma since

$$\int_{\tau^S} r^{\beta-4}h^4 \leq \begin{cases} ch^{\beta+2}, & \text{if } \beta < 2, \\ c|\log h|h^4, & \text{if } \beta = 2 \end{cases} \tag{4.58}$$

and

$$r^{-2}|D(r^2(\tilde{l} - \bar{l}_I))|^2 \leq 8(\tilde{l} - \bar{l}_I)^2 + 2r^2|D(\tilde{l} - \bar{l}_I)|^2. \tag{4.59}$$

□

We conclude

$$\begin{aligned} \|\tilde{l}_I - \tilde{l}_h\|_{L^\infty(\Omega_1)} \leq & ch^{-1}\|\tilde{l}_I - \tilde{l}_h\|_{L^2(\Omega_1)} \\ \leq & ch^{-1}(\|\bar{l}_I - \tilde{l}\|_{L^2(\Omega_1^S)} + \|\tilde{L}_h - \tilde{l}\|_{L^2(\Omega_1^S)}) \\ \leq & c \end{aligned} \tag{4.60}$$

in view of Lemma 4.12 and Remark 4.7 (ii).

Lemma 4.13. *Let $\varphi \in V_h$ and $v = (r^2\varphi)_I \in V_h$ the linear interpolation of $r^2\varphi$ in Ω_1 then*

$$\int_{\Omega_1} r^{-2}|D(r^2\varphi - v)|^2 \leq c \int_{\Omega_1} \varphi^2 \tag{4.61}$$

Proof. For $\tau \in T^1$ we have

$$
\begin{aligned}
|r^2\varphi - v|_{W^{1,\infty}(\tau)} \leq & ch|r^2\varphi|_{W^{2,\infty}(\tau)} \\
\leq & ch\sum_{j=1}^{2} |r^2|_{W^{j,\infty}(\tau)}|\varphi|_{W^{2-j,\infty}(\tau)}
\end{aligned}
\tag{4.62}
$$

because $D^2(\varphi|\tau) = 0$. In view of (4.56) and $r \geq h$ on Ω_1 there holds

$$
|r^2|_{W^{j,\infty}(\tau)} \leq c\inf_{\tau} r^{2-j} \leq c\inf_{\tau} rh^{1-j}
\tag{4.63}
$$

and in view of an inverse estimate

$$
|\varphi|_{W^{2-j,\infty}(\tau)} \leq ch^{j-3}\|\varphi\|_{L^2(\tau)}.
\tag{4.64}
$$

Applying these estimates in (4.62) gives

$$
|r^2\varphi - v|_{W^{1,\infty}(\tau)} \leq ch^{-1}\inf_{\tau} r\|\varphi\|_{L^2(\tau)}
\tag{4.65}
$$

which leads to

$$
\int_{\tau} r^{-2}|D(r^2\varphi - v)|^2 \leq c\int_{\tau}\varphi^2
\tag{4.66}
$$

by estimating the integrand in the L^∞-norm. Summing over $\tau \in T^1$ gives the claim.
$\square$

Lemma 4.14. *Estimate (4.51) holds.*

Proof. For $v_h \in V_h$ we have

$$
a(E, V_h) = O(h^2)\|\tilde{L}_h\|_{H^1(S)}\|V_h\|_{H^1(S)}
\tag{4.67}
$$

and estimate

$$
\begin{aligned}
\int_S \langle DE, D(r^2E)\rangle = & \int_S \langle DE, D(r^2E - V_h)\rangle - \int_S EV_h \\
& + O(h^2)\|\tilde{L}_h\|_{H^1(S)}\|V_h\|_{H^1(S)} \\
\overset{\textit{Lemma } 4.10, \textit{Remark } 4.7}{\leq} & \int_{\Omega_1} \langle DE, D(r^2E - V_h)\rangle + c(h^2 + h|V_h|_{W^{1,\infty}(S\setminus\Omega_1)}) \\
& + \int_S V_h^2 + O(h^2)\|\tilde{L}_h\|_{H^1(S)}\|V_h\|_{H^1(S)}.
\end{aligned}
\tag{4.68}
$$

If v_h interpolates $r^2(\bar{l}_I - \tilde{L}_h)$ in Ω_1 then

$$
\begin{aligned}
\int_{\Omega_1^S} \left\langle DE, D(r^2E - V_h)\right\rangle \leq & \frac{1}{16}\int_S r^2|DE|^2 + 4\int_{\Omega_1^S} r^{-2}|D(r^2E - V_h)|^2 \\
\leq & \frac{1}{16}\int_S r^2|DE|^2 + 8\int_{\Omega_1^S} r^{-2}|D(r^2(\tilde{l} - \bar{l}_I))|^2 \\
& + 8\int_{\Omega_1^S} r^{-2}|D(r^2(\bar{l}_I - \tilde{L}_h) - V_h)|^2 \\
\overset{Lemma\ 4.12(ii), Lemma\ 4.13}{\leq} & \frac{1}{16}\int_S r^2|DE|^2 + ch^2|\log h| + c\int_{\Omega_1^S}(\bar{l}_I - \tilde{L}_h)^2 \\
\leq & \frac{1}{16}\int_S r^2|DE|^2 + ch^2|\log h| \\
& + c\left(\int_{\Omega_1^S}(\bar{l}_I - \tilde{l})^2 + \int_{\Omega_1^S}(\tilde{l} - \tilde{L}_h)^2\right) \\
\overset{Lemma\ 4.12, Remark\ 4.7}{\leq} & \frac{1}{16}\int_S r^2|DE|^2 + ch^2|\log h|.
\end{aligned}
\tag{4.69}
$$

We use (4.69) to estimate the first summand on the right-hand side of (4.68) and obtain

$$
\begin{aligned}
\int_S \left\langle DE, D(r^2E)\right\rangle \leq & ch^2|\log h| + \frac{1}{16}\int_S r^2|DE|^2 + ch|V_h|_{W^{1,\infty}(S\setminus\Omega_1^S)} + \int_S V_h^2 \\
& + O(h^2)\|\tilde{L}_h\|_{H^1(S)}\|V_h\|_{H^1(S)}
\end{aligned}
\tag{4.70}
$$

We estimate v_h with standard interpolation estimates

$$
\begin{aligned}
h|v_h|_{W^{1,\infty}(S_h\setminus\Omega_1)} + h^{-1}\|v_h\|_{L^2(S_h\setminus\Omega_1)} \leq & c\sup_{S\setminus\Omega_1^S}|r^2(\bar{l}_I - \tilde{L}_h)| \\
= & \sup_{\partial\Omega_1^S}|r^2(\bar{l}_I - \tilde{L}_h)| \\
\leq & ch^2
\end{aligned}
\tag{4.71}
$$

where we assume w.l.o.g. that v_h is zero at all nodes in the interior of $S_h \setminus \Omega_1$ and for the last inequality estimate (4.60). Furthermore, we have

$$
\begin{aligned}
\|v_h\|_{L^2(\Omega_1)} \leq & c\|r^2(\bar{l}_I - \tilde{L}_h)\|_{L^2(\Omega_1^S)} \\
\leq & c\|\bar{l}_I - \tilde{l}\|_{L^2(\Omega_1^S)} + \|\tilde{l} - \tilde{L}_h\|_{L^2(\Omega_1^S)} \\
\leq & ch
\end{aligned}
\tag{4.72}
$$

in view of Lemma 4.12 and Remark 4.7 and

$$
\|V_h\|_{H^1(S)} \leq ch^{-1}\|V_h\|_{L^2(S)} \leq ch \tag{4.73}
$$

and

$$
\|\tilde{L}_h\|_{H^1(S)} \leq ch^{-1}\|\tilde{L}_h\|_{L^2(S)} \leq c. \tag{4.74}
$$

□

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